Kinematics of Fluid Flows

Control Mass and Lagrange Flow Description.

- The control mass or the system always consists of the same set of particles of fluid.
- By definition it has a constant mass. The volume and the shape of the control mass can vary such that it contains the same original set of fluid particles.
- System analysis is related to the Lagrange method of flow description.
- The control mass is sometime referred to as the material element of fluid.
- In Lagrange flow description we follow the fluid particle and describe the flow properties (for example velocity, acceleration, pressure…etc) as functions of the particle's initial position at time t= 0.

Control Volume and Eulerian Flow Description.

- The control volume is a prescribed volume in space through which the fluid passes.
- It can be either fixed or moving with a constant speed in a straight line.
- The volume and the shape of the control volume do not vary.
- The control volume is bounded by a control surface.
- The control surface should pass by flow areas of known areas or to be calculated.
- The control volume is related to Eulerian description.

Intensive and Extensive properties

- The intensive fluid or flow property does not depend on the amount of material present in the system.
- Its value is not changed by subdivision of the system.
- Examples of intensive properties: density, Viscosity, pressure, temperature, flow velocity and acceleration.
- The extensive property depends on the amount of material present in the system.
- If a system is subdivided into two sub-systems, the value of the extensive property of the subsystem will be less than the value of the total system.
- Examples of extensive properties: Volume, mass, and momentum

Intensive and Extensive properties

In the present analysis we will consider B any extensive property for a fluid system of mass m_s and b is the corresponding intensive property (property per unit mass of fluid).

The property	Extensive property B	Intensive property b
Mass	m_s in kg	1 kg/kg
Linear momentum	$m_s \bar{v}$ kg m/s	$\bar{v}$ m/s
Angular momentum or moment of momentum	$m_s(\bar{r} \times \bar{v})$ kg m^2/s	$\bar{r} \times \bar{v}$ m^2/s
Kinetic energy	$\frac{1}{2} m_s(\bar{v} \cdot \bar{v})$ kg m^2/s^2	$\frac{\bar{v} \cdot \bar{v}}{2}$ m^2/s^2
Potential energy	$m_s g z$ N m	$g z$ N m/kg
Body force	$\bar{F}$ N	$\frac{f}{m_s}$ m/s^2

Rate of Change of an Extensive Flow Property (Reynolds' Transport Theorem)

$$\frac{DB}{Dt} = \frac{\partial}{\partial t}\iiint_{c.v} \rho b dv + \iint_{c.s} \rho b v.ds$$

This shows that the rate of change of property B for a given set of fluid particles equals the local rate of change of B within the control volume plus the net rate of exchange of B across the control surface.

Note that the inlet flux is negative because the normal component of *q* is opposite to ds and vice versa.

Integral Equation of Mass Conservation.

The equation of mass conservation or continuity equation for a system states that the mass of a given set of particles of fluid does not change. The extensive property B is the mass and b is dimensionless.

Thus, in terms of control volume quantities the mass conservation is,

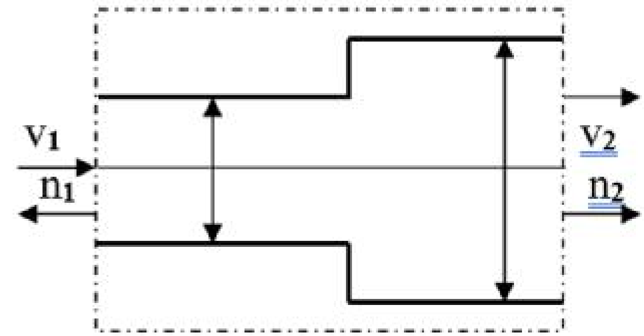
$$\frac{Dm}{Dt} = \frac{\partial}{\partial t}\iiint_{c.v.} \rho dV + \iint_{c.s.} \rho \overline{V}.d\overline{s} = 0$$

Integral form of mass conservation equation

Integral Equation of Mass Conservation. $\frac{Dm}{Dt} = \frac{\partial}{\partial t}\iiint_{c.v.} \rho dV + \iint_{c.s.} \rho \bar{V}.d\bar{s} = 0$

Example:

A steady flow of water through the shown sudden expansion, it is considered that the velocity v_1= 2 m/s. Calculate the velocity v_2 and the mass flow rate.



Solution:

Consider the shown control volume and note that the control surface passes by the flow area and normal to the velocity vector.
The local rate of change of mass within the C.V. is zero. The surface integral gives;

$$-A_1 \rho v_1 + A_2 \rho v_2 = 0$$

and thus, $v_2 = v_1 A_1 / A_2$,

Since the density is constant (Incompressible flow)

$v_2 = 2x\frac{3600}{10000} = 0.72 m/s$ mass flow rate = 1000 x 3.1416 x 0.06 x 0.06 x 2/4 = 5.655 kg / s.

Differential form of mass conservation equation

We would like to derive a differential equation of mass conservation, using Eulerian method of description. The Lagrangian mass conservation will be mentioned without proof.

Consider the convective rate of mass change ie. The surface integral $\iint_{c.s.} \rho v.ds$

and convert it to volume integral using the divergence theorem; $\iint_{c.s.} \rho v.ds = \iint_{c.v.} Div(\rho v)dv$

And then exchange the order of time differentiation and volume integration of the first term (the density is continuous in time and space.)

We get, $\iiint_{c.v.} \frac{\partial \rho}{\partial t} dv + \iiint_{c.v.} Div(\rho v)dv = 0$

Because we consider an arbitrary control volume, we can remove the integration sign dv and equate the integrands, we get, $\frac{\partial \rho}{\partial t} + Div(\rho \vec{V}) = 0$

Expanding the vector form one gets $\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$

Differential form of mass conservation equation

- Where $\vec{V}$ in cartesian coordinates is (u $\vec{i}$, v $\vec{j}$, w $\vec{k}$) and $\vec{i}$, $\vec{j}$, $\vec{k}$ are unit vectors in x, y and z directions respectively. Consider the following special cases:

1. steady compressible flow. The local rate of change of density w. r. t. time is zero $\frac{\partial \rho}{\partial t} = 0$ and thus,

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{(\rho w)}{\partial z} = 0$$

2. Incompressible Flow. Both the local and convective rates of change of density are zero and thus,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

3. Incompressible Two – Dimensional flow. The density is constant and the flow pattern does not change with z coordinate;

$$\frac{\partial w}{\partial z} = 0 \text{ and, } \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Differential form of mass conservation equation

- Example:

Two of three velocity components for an incompressible flow are:
$u = x^2 + 2xz$ and $v = y^2 + 2yz$. What is the most general form of the 3[rd] component?

* Given :-

$u = x^2 + 2xz$, $v = y^2 + 2yz$, incompressible flow

* Required :-

the 3rd component

* solution :-

⇒ Differential form of mass conservation eq.

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}$$

$\frac{\partial \rho}{\partial t}$ → steady

∵ Flow in incompressible :-

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial u}{\partial x} = 2x + 2z , \quad \frac{\partial v}{\partial y} = 2y + 2z$$

$$2x + 2y + 4z + \frac{\partial w}{\partial z} = 0$$

$$\therefore \frac{\partial w}{\partial z} = -2x - 2y - 4z$$

$$\Rightarrow \boxed{w = -2xz - 2yz - 2z^2 + f(x, y)}$$

Differential form of mass conservation equation

- Example:

Given that $u = xy$, $v = 2yz$. Examine whether these velocity components represent two or three-dimensional incompressible flow; if three dimensional, determine the third component.

$\because$ $u = xy$ $\quad v = 2yz$

Firstly, we have to make check $\Rightarrow$ $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = y + 2z$

$\because$ $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \neq 0$ $\Rightarrow$ $\therefore$ there are third component of velocity

to get the 3rd component $\Rightarrow$ $y + 2z + \frac{\partial w}{\partial z} = 0$

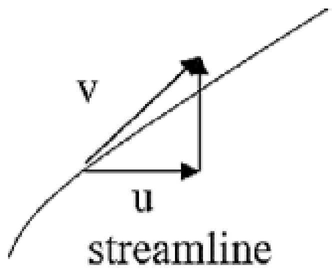
$$\frac{\partial w}{\partial z} = -y - 2z \quad \Rightarrow \boxed{w = -yz - z^2 + F(x, y)}$$

↑ The most general form of the 3rd component

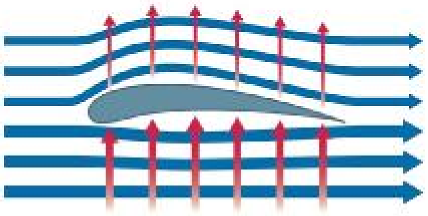
Streamline , Streakline and Pathline

- These are three lines that can be used to describe the fluid motion.
- **Streamline:**

It is a curve in the flow which at any instant is tangent to the velocity vector of the fluid



- There is no flow normal to the streamline
- For unsteady flow the shape of the streamline may change with time
- The streamline equation is thus $\frac{v}{u} = \frac{dy}{dx}$ and $\frac{w}{v} = \frac{dz}{dy}$

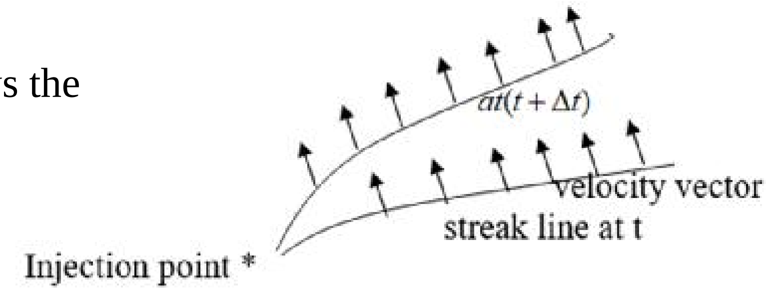


Streamline , Streakline and Pathline

- **Streak line:**

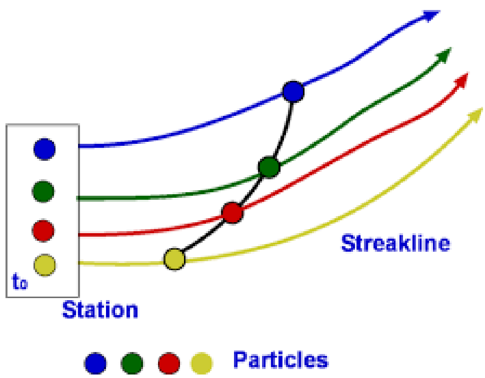
The streak line is the line connecting the present position of fluid particles which previously passed through some point.

- Experimentally continuous smoke or dye injection shows the streak line that passes by the injection point.



- **Path line:**

is the trajectory of a single fluid particle over some period of time in the space.



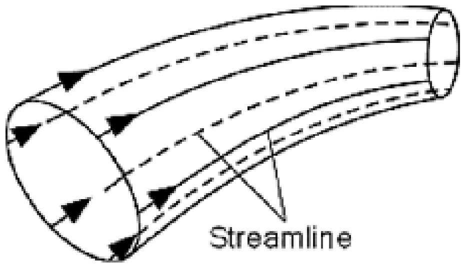
In a steady flow the streamline, streak line and path line are identically the same curve.

Stream function

- **Stream tube:**

A tube formed by streamlines passing by a closed curve in the fluid is called a stream tube.

There is no flow across the sides of the stream tube.



- **The stream Function Ψ**

For cartesian coordinates (two-dimensional flow):

$$u = \frac{\partial \Psi}{\partial y} \text{ And } v = -\frac{\partial \Psi}{\partial x}$$

For cylindrical coordinates:

$$v_r = \frac{1 \partial \Psi}{r \partial \theta} \text{ and } v_\theta = -\frac{\partial \Psi}{\partial r}$$

Example:

- A velocity field given by $V = 2yi + xj$. Drive an expression for acceleration. Find the equation of streamlines. Is the flow incompressible?

$u = 2y \qquad v = x \quad \rightarrow$ 2-Directional, 2-Dimensional flow

compressibility check $\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 + 0 = 0$

$\therefore$ Flow is incompressible

* Acceleration :-

$$a_x = \frac{\partial u}{\partial t} + \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z}\right)$$

$$a_y = \frac{\partial v}{\partial t} + \left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z}\right)$$

$$\therefore a_x = 0 + (2y)(0) + (x)(2) + (0) \Rightarrow a_x = 2x$$

$$a_y = 0 + (2y)(1) + (x)(0) + (0) \Rightarrow a_y = 2y$$

$$\therefore \vec{a} = 2x\,\vec{i} + 2y\,\vec{j}$$

Example continued

$u = 2y \qquad v = 2x$

$\therefore\ u = \frac{\partial \psi}{\partial y} \qquad v = \frac{-\partial \psi}{\partial x}$

$\therefore\ 2y = \frac{\partial \psi}{\partial y}$

$\psi = y^2 + F(x)$

now differentiate ψ with respect to x

$\frac{\partial \psi}{\partial x} = 0 + F'(x) = -v$

$F'(x) = -2x$

$F(x) = -x^2 + C$

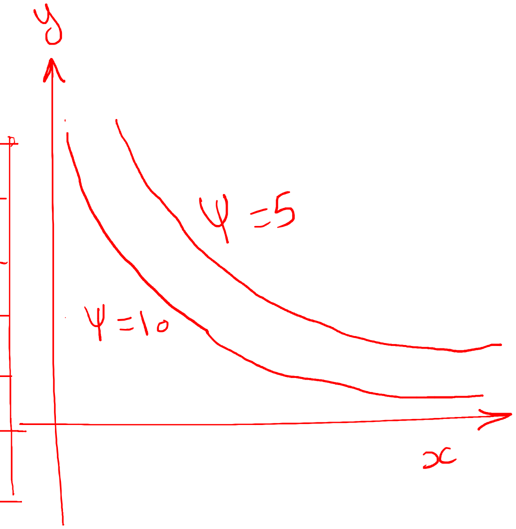
$\therefore\ \psi = y^2 - x^2 + C$

to sketch stream lines :-

assume $\psi = 10$

$\therefore\ y = \sqrt{x^2 + 10}$

x \ ψ	10	5
1	3.3	
2	3.74	
3	4.35	
0.5	3.2	
0.25	3.17	



Vorticity and circulation

- The vorticity is defined as the curl of the velocity vector $\vec{W} = \nabla \times \vec{V}$ It is a measure of the local rotation of the fluid.
- The circulation is the line integral on a closed curve in the fluid of V. dL with dL = (dx i, dy j, dz k);

$$G = \oint \vec{V} . \vec{dL}$$

The vorticity $\vec{\Omega}$ equals twice the angular velocity of the fluid. This can be shown for fluid with solid body rotation $V_0 = r\varpi$ where w is the angular velocity in rad /s i.e., $\Omega = 2.\varpi$

The flow of zero vorticity is known to be irrotational or potential flow $u = \frac{\partial \phi}{\partial x}$, $v = \frac{\partial \phi}{\partial y}$; $w = \frac{\partial \phi}{\partial z}$

since, ϕ : potential Function (in the case of irrotational flow)

$$\vec{\Omega} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = 0 \qquad \vec{\Omega} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} - \left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right)\vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k} = 0$$

for 2-Dimensional flow $\Rightarrow \vec{\Omega} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$

the condition of irrotational flow